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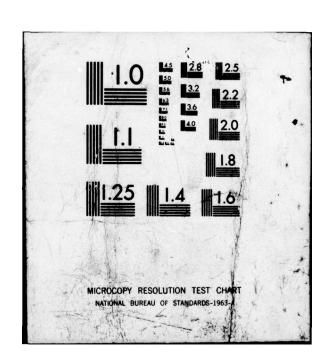
UNIFIED TREATMENT OF INEQUALITIES OF THE
WEIERSTRASS PRODUCT TYPE

by

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ABSTRACT

A typical inequality of the Weierstrass product type states that

$$\prod_{i=1}^{n} (1+A_i) \geq (n+1)^n \prod_{i=1}^{n} A_i, \text{ where } A_i \geq 0, \; i=1,\ldots,n, \text{ and } \sum_{i=1}^{n} A_i = 1.$$

This short note shows that the powerful tools of majorization and Schur functions provide a unified method for deriving a variety of Weierstrass type inequalities.

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In a note by Klamkin and Newman [1], it was shown that

$$(1) \qquad \prod_{i=1}^{n} (1+A_i) \geq (n+1)^n \prod_{i=1}^{n} A_i$$

$$(2) \qquad \prod_{i=1}^{n} (1-A_i) \geq (n-1)^n \prod_{i=1}^{n} A_i,$$

where $A_i \geq 0$ $(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} A_i = 1$. Under the same conditions it
was shown by Klamkin in [2] that

$$(3) \qquad \frac{\prod_{i=1}^{n} (1+A_i)}{(n+1)^n} \geq \frac{\prod_{i=1}^{n} (1-A_i)}{(n-1)^n}$$

with equality if $A_i = \frac{1}{n}$. In [2] the author refers to a similar inequality
to (2) Ky Fan [[4], p. 363] under tighter conditions on A_i but more relaxed
condition on ΣA_i, namely,

$$(4) \qquad \prod_{i=1}^{n} (1-A_i) \geq \{ \frac{n - \Sigma A_i}{\Sigma A_i} \}^n \prod_{i=1}^{n} A_i \qquad \text{for } 0 < A_i \leq \frac{1}{2}.$$

Inequalities (1), (2) and (3) are proved by the authors as extensions of
the Weierstrass product inequalities [see [1]].

The main purpose of this note is to show that the inequalities (1), (2), (3) and (4) can be obtained by a uniform approach using the powerful tools of majorization and Schur-functions. Majorization (defined in Section 2) is a partial ordering in R_n, the n-dimensional Euclidean space. A Schur-function is a function that is monotone with respect to this partial ordering. In Section 2 we show that inequalities (1), (2), (3) and (4) follow immediately by observing that certain functions are Schur-functions, thus providing a unifying and more transparent proof of these inequalities.

<u>Section 2</u>.   We now give the standard definitions and results of majorization and Schur-functions needed for proving inequalities (1), (2), (3) and (4).

Given a vector $x = (x_1, \ldots, x_n)$, let $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$ denote a decreasing rearrangement of $x_1, \ldots x_n$.

<u>Definition 1</u>.   A vector $\underset{\sim}{x}$ is said to majorize a vector $\underset{\sim}{x}'$ if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} x'_{[i]}, \quad j=1, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} x'_{[i]};$$

in symbols $\underset{\sim}{x} \overset{m}{\geq} \underset{\sim}{x}'$.

<u>Definition 2</u>.   A function $f: R_n \to R$ is said to be a <u>Schur-convex</u> (<u>Schur-concave</u>) function if $\underset{\sim}{x} \overset{m}{\geq} \underset{\sim}{x}'$ implies that $f(\underset{\sim}{x}) \geq (\leq) f(\underset{\sim}{x}')$. Functions which are either Schur-convex or Schur-concave are called <u>Schur-functions</u>.

<u>Theorem 1</u>.   Let $\phi: R \to R$ be a log-convex (log-concave) real-valued function. Let $g: R_n \to R$ defined by $g(\underset{\sim}{x}) = \prod_{i=1}^{n} \phi(x_i)$, where $\underset{\sim}{x} = (x_1, \ldots, x_n)$. Then g is Schur-convex (Schur-concave).

<u>Theorem 1</u> follows immediately from a theorem by Ostrowski, (1952) [3].

We are now ready to prove inequalities (1), (2), (3) and (4). Let

$\underset{\sim}{A} = (A_1, \ldots, A_n)$, where $A_i \geq 0$, $\sum_{i=1}^{n} A_i = 1$. Clearly $\underset{\sim}{A} \overset{m}{\succeq} (\frac{1}{n}, \ldots, \frac{1}{n})$.

Now let $\phi_1(x) = \frac{1+x}{x}$, where $x \geq 0$. Then $\phi_1(x)$ is log-convex. Therefore

by Th. 1, $\pi_{i=1}^{n} \phi_1(x_i)$ is Schur-convex. Hence $\pi_{i=1}^{n} \frac{(1+A_i)}{A_i} \geq (n+1)^n$ which

establishes (1).

Now let $\phi_2(x) = \frac{1+x}{1-x}$, $0 \leq x < 1$. Then $\phi_2(x)$ is log-convex and so by

Theorem 1, $\pi_{i=1}^{n} \phi_2(x_i)$ is Schur-convex. It then follows that $\pi_{i=1}^{n} \frac{(1+A_i)}{(1-A_i)} \geq \frac{(n+1)^n}{(n-1)^n}$,

which establishes (3). Equality holds if $A_i = \frac{1}{n}$ since $\phi_2(x)$ is strictly

log convex.

Now let $\phi_3(x) = \frac{1-x}{x}$; then $\phi_3(x)$ is log-convex for $0 < x \leq \frac{1}{2}$ and log-concave

for $\frac{1}{2} \leq x < 1$. Also let $\underset{\sim}{A} = (A_1, \ldots, A_n)$, where $0 \leq A_i \leq \frac{1}{2}$. Clearly

$\underset{\sim}{A} \overset{m}{\geq} (\frac{\sum A_i}{n}, \ldots, \frac{\sum A_i}{n})$, so that $\pi_{i=1}^{n} (\frac{1-A_i}{A_i}) \geq [\frac{1-\sum A_i/n}{\sum A_i/n}]^n = [\frac{n - \sum A_i}{\sum A_i}]^n$, establishing (4).

Finally, let $\underset{\sim}{A}^1 = ((n-1)A_1, \ldots, (n-1)A_n)$ and let $\underset{\sim}{A}^2 = (1-A_1, \ldots, 1-A_n)$ where

$A_i \geq 0$, $\sum A_i = 1$. It can be easily verified that $\underset{\sim}{A}^1 \overset{m}{\geq} \underset{\sim}{A}^2$. By Theorem 1,

$g(\underset{\sim}{x}) = \pi_{i=1}^{n} x_i$ is a Schur-concave function. It then follows that

$$(n-1)^n \pi_{i=1}^{n} A_i = \pi_{i=1}^{n} (n-1)A_i \leq \pi_{i=1}^{n} (1-A_i),$$

proving inequality (2).

It is apparent that many additional inequalities of the Weierstrass product

type can be formulated and proved by choosing the appropriate log-concave function,

forming products to obtain a Schur-convex function, and then using Definition 2 above.

References

M.S. Klamkin, D. J. Newman. Extensions of the Weierstrass product inequalities. Math. Mag., 43 (1970), 137-140.

M.S. Klamkin. Extensions of the Weierstrass product inequalities II. Amer. Math. Monthly 82 (1975), No. 7, 741-742.

A. Ostrowski. Sur quelques applications des functions convexes et concaves au sens de I. Schur. J. Math. Pure. Appl. 31, (1952), 253-292.

D. S. Mitrinovic. Analytic Inequalities. Springer-Verlag, Heidelberg (1970).

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A typical inequality of the Weierstrass product type states that
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